Examining these equations, we see that they are all of the type:

$$S_n=\frac{n(n+1)}{2}.$$

By taking a few additional ones, we find that the formula continues to hold. Thus:

$$S_6=1+2+3+4+5+6=21=\frac{6.7}{2}.$$

Trial of additional instances increases the "moral" certainty that the formula is true for every positive integer n, but no matter how many instances we may have the patience to try, mathematical certainty is not achieved thereby. This is attained by means of the mind's power of operating with mathematical certainty upon unspecified numbers.

The invariable method of mathematical induction is to prove that whenever the formula in question holds in any particular instance, it also holds in the next following instance.*

Recurring to our example, we must show that if the formula $S_n=\frac{n(n+1)}{2}$ holds for any particular value of n, say $n=k$, it also holds for the next value of n or $k+1$.

(*A*) That is, we must show that, whenever $S_k=\frac{k(k+1)}{2}$ holds, then, $S_{k+1}=\frac{(k+1)(k+2)}{2}$ holds also.

Proof. By definition, $S_{k+1}=S_k+k+1$.

Substituting the assumed value of S_k,

$$S_{k+1}=\frac{k(k+1)}{2}+k+1=\frac{(k+1)(k+2)}{2}.$$

The proof required above is thus made, but the result is purely hypothetical. We pass to actually valid results as follows.

(*B*) Trial has already shown that the formula is true for S_1 to S_6. Since the formula is true for S_6, we know, without trial but solely by the proof at (*A*), that the formula holds for S_7. since it is true for S_7, we know similarly without trial, that it is true for S_8. Similarly we know that the formula is true for S_9, for S_{10}, and so on to any S_n whatever.

The success of the proof hinges upon three things:

1. The ability to express any instance in terms of the next preceding, even when the latter is not specified.

*The application of the method presupposes that the cases to be considered have in some way been arranged in a definite order and numbered consecutively, and that after each case follows another.

2. The ability to make the proof (A) without specifying the particular instance that is being considered.

3. The power of the mind to see with certainty that repetition of the steps at (B) would lead to any given n, without actually going through all these steps.

These three things are but various phases of the mind's power of operating with certainty upon unspecified numbers.

In any proof by mathematical induction, the following two parts must unfailingly be present:

1). The proof that *if* the statement holds in any particular instance, it also holds in the next, and

2). The proof that the statement actually holds in the first instance (that is, in *some* particular case).

These two proofs are quite independent and may therefore be made in either order, but the relation to true induction is most apparent when the formula to be proved is actually discovered by true induction (as in the example above), thus amply making the second proof at the outset.

Beginners are prone to regard one or the other of these proofs as sufficient in itself. The need of both may be made clear by non-mathematical illustrations and by mathematical examples.

As a non-mathematical illustration we may consider a row of bricks so arranged that whenever any brick is knocked over, it will in its fall knock over its neighbor on the right. But this is only potential. In order actually to knock over the whole row, it is necessary and sufficient actually to knock over the first brick. If this cannot be done the whole row cannot be knocked over.

A second illustration is that of a ladder.* "We must have a ladder by which to climb from any round (the kth) to the next round (the $k+1$st); but the ladder must rest on a solid basis so that we can get on to the ladder (the $k=1$ or $k=2$ rounds)."

Mathematical examples can also be given in which one of the parts can be proved but not the other, and hence the statement in question is not proved.

1. Consider the series,

$$1.1!,\ 2.2!,\ 3.3!,\ \ldots.$$

It is readily proved that if the sum of the first k terms is $(k+1)!$, then the sum of the first $k+1$ terms is $(k+1+1)!$. Or, denoting the sum of the first n terms of any series by S_n, if, for this series, the formula $S_n=(n+1)!$ holds for any particular value of n, it also holds for the next following value of n. But there is no value of n for which it can be proved to hold, and therefore the formula is not proved. (If it could be proved for any partic-

*Dickson, *College Algebra*, p. 100.

ular value of n after the first, the formula would of course be proved from that value of n on.)

2. Considering the series,

$$1, \tfrac{3}{2}, \tfrac{5}{4}, \tfrac{7}{8}, \tfrac{9}{16}, \ldots.$$

it can be proved that if the formula

$$S_n = 1 - \frac{2n+3}{2^{n-1}}$$

holds for any particular value of k, it also holds for $k+1$. But no value k can be found for which the formula holds and hence it is not proved.

On the other hand it may be possible to verify a certain statement in many consecutive instances, without leading to the conclusion that it is true in every instance.

1. The expression, $1-2^{n+2}+2.3^{n+1}-4^{n+1}+5^n$ is zero for $n=0, 1, 2, 3,$ but not zero for $n=4$.

2. $(4^n-6.3^n+14.2^n-14)[1-(-1)^n]$ is zero for $n=0, 1, \ldots, 6,$ but not zero for $n=7$.

3. n^2+n+17 is prime for $n=0, 1, 2, \ldots, 16,$ but not for $n-17$.

4. $2n^2+29$ is prime for $n=0, 1, 2, \ldots, 28,$ but not for $n=29$.

5. n^2-n+41 is prime for $n=0, 1, 2, \ldots, 40,$ but not for $n=41$.

6. The theorem that if an odd prime be increased by 3, the result is the product of an odd prime and a power of two, holds for the odd primes to 43, but does not hold for 47.

III. Exercises.

In the exercises that follow, a formula for S_n is to be proved, unless otherwise specified. The finding of the formula by true induction is one of the most fascinating activities, and offers, in proportion to the difficulty of the task, more or less opportunity for the application of mathematical ability and the enjoyment of mathematical inspiration. Not to rob the reader of the possibility of the pleasures, the answers are not as a rule given with the exercises, but are collected at the end.

The exercises are not arranged in order of difficulty. This will naturally vary with different minds. Of those whose answers are reserved, the following are among the easiest, arranged somewhat in order of supposed difficulty: 21, 10, 13, 1, 32, 23, 12, 11, 4, 9. The degree of difficulty of the exercises may be diminished by giving a part of the result. Thus, No. 3 becomes very easy if it is given that $4n^2+6n-1$ is a factor of the result, and No. 7 becomes moderately easy if it is given that $2n+1$ is a factor of the result.

In each of the following the sum of the series is to be found by mathematical induction.

(1) $1.2, 2.3, 3.4, \ldots$ (5) $1^2, 2^2, 3^2, \ldots$

(2) $1.2, 3.4, 5.6, \ldots$ (6) $1^2, 3^2, 5^2, \ldots$

(3) $1.3, 3.5, 5.7, \ldots$ (7) $2^2, 4^2, 6^2, \ldots$

(4) $2.3, 4.6, 6.8, \ldots$ (8) $4^2, 7^2, 10^2, 13^2, \ldots$

(9) $0, 2.3, 3.8, 4.15, \ldots, k(k^2-1), \ldots$

(10) $1^3, 2^3, 3^3, \ldots$

(11) $1^3, 3^3, 5^3, \ldots$

(12) $2^3, 4^3, 6^3, \ldots$

(13) $2.3.4, 3.4.5, 4.5.6, \ldots$

(14) $1.3.5, 3.5.7, 5.7.9, \ldots$

(15) $1.1^2, 2.3^2, 3.5^2, 4.7^2, \ldots$

(16) $1.2^2, 2.3^2, 3.4^2, 4.5^2, \ldots$

(17) $1, 4, 10, 20, \ldots, \dfrac{k(k+1)(k+2)}{3!}, \ldots$

(18) $1^4, 2^4, 3^4, 4^4, 5^4, \ldots$

(19) $1.2.3.4, 2.3.4.5, 3.4.5.6, \ldots$

(20) $1, 5, 15, 35, \ldots, \dfrac{k(k+1)(k+2)(k+3)}{4!}, \ldots$

(21) $\dfrac{1}{1.2}, \dfrac{1}{2.3}, \dfrac{1}{3.4}, \cdots$

(22) $\dfrac{1}{1.3}, \dfrac{1}{2.4}, \dfrac{1}{3.5}, \cdots$

(23) $\dfrac{1}{3.5}, \dfrac{1}{5.7}, \dfrac{1}{7.9}, \cdots$

(24) $\dfrac{1}{1.2.3}, \dfrac{1}{2.3.4}, \dfrac{1}{3.4.5}, \cdots$

(25) $\dfrac{1}{1.2.3}, \dfrac{3}{2.3.4}, \dfrac{5}{3.4.5}, \cdots$

(26) $\dfrac{1}{2.3.4}, \dfrac{2}{3.4.5}, \dfrac{3}{4.5.6}, \cdots$

(27) $\dfrac{1}{1.3.5}, \dfrac{1}{3.5.7}, \dfrac{1}{5.7.9}, \cdots$

(28) $\dfrac{1}{2.5.8}, \dfrac{2}{5.8.11}, \dfrac{3}{8.11.14}, \cdots$

(29) $\dfrac{1}{1.2.3.4}, \dfrac{1}{2.3.4.5}, \dfrac{1}{3.4.5.6}, \cdots$

(30) $1^5, 2^5, 3^5, 4^5, \ldots$

(31) $1^6, 2^6, 3^6, 4^6, \ldots$*

(32) $1, 2.2, 3.2^2, 4.2^3, \ldots$

(33) $1, 2.3, 3.3^3, 4.3^3, \ldots$

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*On the series $1^n, 2^n, 3^n, \ldots$ see Chrystal's *Alpha*, Vol. I, p. 486.

(34) $1, 2.5, 3.5^2, 4.5^3, \ldots$

(35) $1, 3.2, 5.2^2, 7.2^2, \ldots$

(36) $1, 4.3, 7.3^2, 10.3^3, \ldots$

(37) $1, 5.4, 9.4^2, 13.4^3, \ldots$

(38) $1, 4.2, 9.2^2, 16.2^3, \ldots$

(39) $1, 4.3, 9.3^2, 16.3^3, \ldots$

(40) $1, 8.2, 27.2^2, 64.2^3, \ldots$

(41) $1, 3.2, 6.2^2, 10.2^3, \ldots, \frac{k(k+1)}{2}2^{k-1}, \ldots$

(42) $1, \frac{4}{3}, \frac{9}{3^2}, \frac{16}{3^3}, \ldots$

(43) $a, 2(a+1), 3(a+2), 4(a+3), \ldots$

Show that:

(44) $2.6.10.14, \ldots, (4n-6)(4n-2)=(n+1)(n-2), \ldots, (2n-1)2n.$

(45) $n(n+1)(n+2), \ldots, (2n-2)=1.3.5.7, \ldots, (2n-3).2^{n-1}.$

(46) $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^n})=1+x+x^2+\ldots+x^{2^{n+1}-1}$

(47) Show that $x^{2n}-y^{2n}$ is divisible by $x+y$.

(48) Show that $n^{13}-n$ is divisible by 13, for every positive integer n.

(49) If p is a prime number, show that n^p-n is divisible by p for every positive integer n.

(50) Show that $2.7^n+3.5^n-5$ is divisible by 24 for every positive integer n.

Suggestion: Assume, 1) $2.7^k+3.5^k-5$=multiple of 24.
To show, $2.7^{k+1}+3.5^{k+1}$=multiple of 24.
Multiply 1) by 7, $2.7^{k+1}+3.5^k.7-5.7$=multiple of 24.
Hence, $2.7^{k+1}+3.5^{k+1}-5-30+3.5.2^k$=multiple of 24.
To prove the last assertion, we must show that 6.5^k-30=multiple of 24, which is easily done.

(51) Show that $3.5^{2n+1}+2^{3n+1}$ is divisible by 17 for every positive integer n.

(52) If n is a positive integer, show that $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}a^{n-k}b^k+\ldots$

(53) If all the positive integers of n and fewer digits be written, show that the number of times any digit (other than zero) occurs is $n.10^{n-1}$.

(54) If the positive integers are grouped as follows: [(1, 2), (3)]; [(4, 5, 6), (7, 8)]; [(9, 10, 11, 12), (13, 14, 15)]; ... and these groups taken in pairs as indicated, prove that the sum of the numbers in the two groups of any pair is the same.

(55) $\sum_{k=1}^{n}(a+kb)^2=n[a^2+ab(n+1)+\frac{b^2}{6}(2n^2+3n+1)].$

(56) $\sum_{k=1}^{n}\frac{(2k)!}{k!\,2^k}=1.3.5\ldots(2n-1).$

(57) Letting t_k denote the kth term of a series, show that, if $t_k=k(k+1)\dots(k+q-2)$, $S_n=\dfrac{n(n+1)\dots(n+q-1)}{q}$.

(58) If $t_k=\dfrac{1}{k(k+1)\dots(k+p)}$, $S_n=\dfrac{1}{p}\left[\dfrac{1}{p!}-\dfrac{1}{(n+1)\dots(n+p)}\right]$.

(59) Assuming that the formulas,

$$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y,$$
$$\cos(x\pm y)=\cos x\cos y\pm\sin x\sin y,$$

have been proved for all acute angles x, y, prove that they hold for all posi- angles x, y. (In this proof, the quadrants are to be regarded as numbered consecutively, 5, 6, 7, 8, etc., as the angle increases beyond 360°, and the formulas of the type $\sin(90+x)=\cos x$, are to be accepted as proved for every angle x.)

(60) $(\cos a+i\sin a)^n=\cos na+i\sin na$. DeMoivre's formula.

Sum the following:

(61) $\sin a$, $\sin 2a$, $\sin 3a$, ...

(62) $\sin a$, $\sin 3a$, $\sin 5a$, ...

(63) $\cos a$, $\cos 2a$, $\cos 3a$, ...

(64) $\cos a$, $\cos 3a$, $\cos 5a$, ...

IV. Answers.

(1) $\dfrac{n(n+1)(n+2)}{3}$.

(2) $\dfrac{n(n+1)(4n+1)}{3}$.

(3) $\dfrac{n(4n^2+6n-1)}{3}$.

(4) $\dfrac{4n(n+1)(n+2)}{3}$.

(5) $\dfrac{n(n+1)(2n+1)}{6}$.

(6) $\dfrac{n(2n-1)(2n+1)}{6}$.

(7) $\dfrac{2n(n+1)(2n+1)}{3}$.

(8) $\dfrac{n(6n^2+15n+11)}{2}$.

(9) $\dfrac{(n-1)n(n+1)(n+2)}{4}$.

(10) $\left[\dfrac{n(n+1)}{2}\right]^2$.

(11) $n^2(2n^2-1)$.

(12) $2n^2(n+1)^2$.

(13) $\dfrac{(n+1)(n+2)(n+3)}{3}$.

(14) $n(2n^3+8n^2+7n-2)$.

(15) $\dfrac{n(n+1)(6n^2-2n-1)}{6}$.

(16) $\dfrac{n(n+1)(n+2)(3n+5)}{12}$.

(17) $\dfrac{(n+3)!}{4!(n-1)!}$.

(18) $\dfrac{n(n+1)(6n^3+9n^2+n-1)}{30}$.

(19) $\dfrac{n(n+1)(n+2)(n+3)(n+4)}{5}$.

(20) $\dfrac{(n+4)!}{5!(n-1)!}$.

(21) $\dfrac{n}{n+1}$.

(22) $\dfrac{n(3n+5)}{4(n+1)(n+2)}$.

(23) $\dfrac{n}{3(2n+3)}$.

(24) $\frac{1}{2}\left[\frac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right]$.

(25) $\dfrac{n(3n+1)}{4(n+1)(n+2)}$.

(26) $\dfrac{n(n+1)}{4(n+2)(n+3)}$.

(27) $\dfrac{n(n+2)}{3(2n+1)(2n+3)}$.

(28) $\dfrac{n(n+1)}{4(3n+2)(3n+5)}$.

(29) $\dfrac{1}{3}\left[\dfrac{1}{3!}-\dfrac{1}{(n+1)(n+2)(n+3)}\right]$.

(30) $\dfrac{n^2(n+1)^2(2n^2+2n-1)}{3.4}$.

(31) $\dfrac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{6.7}$.

(32) $(n-1)2^n+1$.

(33) $\dfrac{(2n-1)3^n+1}{4}$.

(34) $\dfrac{(4n-1)5^n+1}{16}$.

(35) $(2n-3)2^n+3$.

(36) $\dfrac{(6n-7)3^n+7}{4}$.

(37) $\dfrac{(12n-13)4^n+13}{9}$.

(38) $(n^2-2n+3)2^n-3$.

(39) $\dfrac{(n^2-n+1)3^n-1}{2}$.

(40) $[(n-1)^3-6(n-2)]2^n+13$.

(41) $2^{n-1}(n^2-n+2)-1$.

(42) $\dfrac{3^{n+1}-(n^2+3n+3)}{2.3^{n-1}}$.

(43) $\dfrac{n(n+1)(3a+2n-2)}{6}$.

(61) $\dfrac{\sin\frac{na}{2}\sin\frac{(n+1)a}{2}}{\sin\frac{a}{2}}$.

(62) $\dfrac{\sin^2 na}{\sin a}$.

(63) $\dfrac{\sin\frac{na}{2}\cos\frac{(n+1)a}{2}}{\sin\frac{a}{2}}$.

(64) $\dfrac{\sin 2na}{2\sin a}$.

ON THE GENERAL TANGENT TO PLANE CURVES.*

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The object of this note is to work out without the Calculus a certain well known formula for a tangent to a plane algebraic curve at an ordinary (not a singular) point, and especially to show how this result is easily extended to the loci of transcendental equations. The formula found is readily developed by aid of the Differential Calculus, but is here found by other means. It might therefore be used in a course which does not presuppose the Calculus.

Let us take, in rectangular Cartesian coordinates, the general equation of a proper nth degree locus in the form

*Read before the Chicago Section of the American Mathematical Society, April 18, 1908.

[1] $$a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+\ldots+\sum_{s=0}^{s=n} a_{n-s,\,s}\,x^{n-s}y^s=0.$$

Let $[\alpha, \beta]$ be an ordinary point of the curve, so that at this point only one tangent to the curve can be drawn. Transform the equation by the substitution

[2] $$x=x'+\alpha, \quad y=y'+\beta.$$

The new origin will be on the curve; and therefore the independent term vanishes. The resulting equation takes the form

[3] $$a_{10}x'+a_{01}y'+2a_{20}\alpha\,x'+a_{11}\beta\,x'+a_{11}\alpha\,y'+2a_{02}\beta\,y'+\ldots$$
$$+\sum_{s=0}^{s=n} a_{n-s,\,s}\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\}$$
$$+\text{terms of second degree and higher in } x',\ y'=0.$$

Let $u^{(p)}$ be identically equal to the sum of all the terms of degree p in x', y' in this equation. Then the equation takes the form

[3$_a$] $$u^{(1)}+u^{(2)}+u^{(3)}+\ldots+u^{(n)}=0.$$

Then is $u^{(1)}=0$ the equation of the tangent to this curve at the origin. (See Salmon-Fiedler, *Hoeheren ebenen Kurven*, 2nd ed., p. 31.) For since the origin is now an ordinary point of the curve, $u^{(1)}$ is not identically equal to zero. [If the origin were a double point $u^{(1)}$ would be identically equal to zero. Then in order to find the equations of the tangents it would be necessary to deal with $u^{(\sigma)}$, where $u^{(\sigma)}$ is the first of the quantities $u^{(2)}$, $u^{(3)}$, ... which is not identically zero. See Salmon-Fiedler, *l. c.* This case is excluded from the present discussion by the first assumption that (α, β) is an ordinary point of the locus of [1]].

Now from [3] it is easy to write

$$u^{(1)}\equiv\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\}=0;$$

or,

$$\frac{y'}{x'}=-\frac{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}[n-s]\alpha^{n-s-1}\beta^s}{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

as the equation of the tangent to the locus of [3] at the origin. Hence by [2], the tangent to the locus of [1] at the point $[\alpha, \beta]$ is

$$[4] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,s}\,s\,\alpha^{n-s}\beta^{s-1}}.$$

EXAMPLE 1. Applying this to the circle $x^2+y^2=r^2$, we have for its tangent at $[\alpha, \beta]$

$$\frac{y-\beta}{x-\alpha} = -\frac{2\alpha}{2\beta} = -\frac{\alpha}{\beta}.$$

Since $\alpha^2+\beta^2=r^2$ this reduces to the usual formula

$$x\alpha+y\beta=r^2.$$

EXAMPLE 2. In the conic $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+a_{00}=0$, we have

$$\frac{y-\beta}{x-\alpha} = -\frac{2a_{20}\alpha+a_{11}\beta+a_{10}}{2a_{02}\beta+a_{11}\alpha+a_{01}}$$

for the tangent at $[\alpha, \beta]$. Since $[\alpha, \beta]$ is on the locus, it is easy to reduce this equation to the form

$$[2a_{20}\alpha+a_{11}\beta+a_{10}]x+[2a_{02}\beta+a_{11}\alpha+a_{01}]y+a_{10}\alpha+a_{01}\beta+2a_{00}=0,$$

the general equation of a tangent to a conic.

Now, in the equation

$$y-\sin x\cos y=0$$

suppose that $\sin x$ and $\cos y$ are replaced by their expansions in series and that the product is taken. The resulting equation with infinite number of terms, may evidently be dealt with by the same method as that which we have used with reference to [1]. And so of the locus of any such equation. Instead of formula [4] it is evident that we should in these cases have the tangent at $[\alpha, \beta]$ determined by the following equation:

$$[5] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum_{n=1}^{n=\infty}\sum_{s=0}^{s=n} a_{n-s,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum_{n=1}^{n=\infty}\sum_{s=0}^{s=n} a_{n-s,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

provided both infinite series are convergent.

If the equation of the given locus is $y-f(x)=0$ where f denotes an algebraic function of x or an infinite convergent series of terms containing x only in positive integral powers and if the coefficients are represented by the same quantities as before; then equations [4] and [5] become

$$[6] \qquad \frac{y-\beta}{x-\alpha}=-\sum_{n=1}^{n=n} a_{n,0}\, n\, \alpha^{n-1}$$

where the range of n is finite or infinite according as $f(x)$ is or is not an algebraic function of x. In the latter case, of course, the series of [6] must be convergent for the given value of α in order that the formula may be employed.

It may be pointed out that formulae [4], [5], [6] can be obtained by differentiation and the substitution of α, β in the results. Thus if $u=0$ is the equation of the curve, these become, respectively,

$$\frac{y-\beta}{x-\alpha}=-\frac{D_x u}{D_y u};\quad \frac{y-\beta}{x-\alpha}=-\frac{D_x u}{D_y u};\quad \frac{y-\beta}{x-\alpha}=-D_x u,$$

where x and y are replaced by α and β in every $D_x u$ and $D_y u$. Hence, the interest which attaches to the discussion in this note is not in the results themselves but in the method of obtaining them without the aid of the Calculus.

EXAMPLE 3. What is the tangent to the curve $y=\sin x$ at the point $[\alpha, \beta]$?

We suppose that this problem is assigned to a pupil who is yet unacquainted with the Calculus, but one who knows from Trigonometry the series

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots,$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots.$$

The given equation becomes

$$y=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots,$$

and therefore by [6] the equation to the tangent is

$$\frac{y-\beta}{x-\alpha}=1-\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}-\ldots$$

But the series of the second member is equal to $\cos\alpha$. Hence for the required tangent we have finally the equation

$$y=\beta+[x-\alpha]\cos\alpha.$$

EXAMPLE 4. Similarly, the tangent to $y=\cos\alpha$ at the point $[\alpha, \beta]$ is

$$y=\beta-[x-\alpha]\sin\alpha.$$

EXAMPLE 5. What is the tangent to $y=\tan x$ at the point $[\alpha, \beta]$?

EXAMPLE 6, What is the tangent to $y=\log_e[1+x]$ at the point $[\alpha, \beta]$? We have

$$y=\log_e[1+x]=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots, \quad |x|<1;$$

and therefore [6] yields

$$\frac{y-\beta}{x-\alpha}=1-\alpha+\alpha^2-\alpha^3+\ldots=\frac{1}{1+\alpha}, \text{ since } |\alpha|<1.$$

Thus, for the required equation we have

$$y=[1+\alpha]x+\beta-\alpha^2.$$

EXAMPLE 7. Similarly, the tangent to $y=\log_e[1-x]$ at the point $[\alpha, \beta]$ is

$$y=[1-\alpha]x+\beta+\alpha^2.$$

[Remark. The chief difficulty in using the foregoing method with pupils beginning the study of Analytics will be in the discussion following equation $[3_a]$ above. A reference to Salmon (or Salmon-Fiedler, *l. c.*) will suffice to show that the difficulty is by no means insurmountable. If the tangent is defined as a straight line which passes through two consecutive points, the method by which Salmon shows that our $u^{(1)}=0$ is the tangent to our $u^{(1)}+u^{(2)}+\ldots+u^{(n)}=0$ will be easily within the grasp of the earnest student. Nothing else in the paper presents any intrinsic difficulty whatever. And the bright student will undoubtedly take interest in a simple general method and formula for solving numerous special problems which are continually arising.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

46. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

A borrows \$500 from a building and loan association and agrees to pay \$9.50 per month for 72 months, the first payment to be made at the end of the first month. What rate of interest does he pay? The association claims to charge only 8% (the legal rate in Alabama). How can the per cent. be figured out?

Solution of unsolved problem in Vol. II, p. 74, by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let r=rate per month, $12r$=rate per annum, p=sum borrowed, n=number of payments, q=cash payment. Then, from Algebra, we get

$$q=\frac{pr(1+r)^n}{(1+r)^n-1},\quad q=9\tfrac{1}{2},\quad p=\$500,\quad n=72.$$

$$\therefore (q-pr)(1+r)^n=q, \text{ and } (19-1000r)(1+r)^{72}=19.$$
$$\therefore r=.00911, \text{ and } 12r=.10932=10.932\% \text{ per annum.}$$

ALGEBRA.

293. Proposed by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

Prove by mathematical induction that $\frac{(x-a)^{m-1}}{(m-1)!}f^{m-1}(a)+\frac{(x-a)^{m-2}}{(m-2)!}+\ldots.+\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a)$ will be the remainder when $f(x)$ is divided by $(x-a)^m$.

Solution by the PROPOSER.

If $\phi(x)$ be the quotient found by dividing $f(x)$ by $(x-a)$ we can write the identity

$$\frac{f(x)}{x-a}=\phi(x)+\frac{f(a)}{x-a}.$$

Differentiating both members and solving for $f(x)/(x-a)^2$,

$$\frac{f(x)}{(x-a)^2}=\frac{f'(x)}{x-a}+\frac{f(a)}{(x-a)^2}-\phi'(x). \qquad (1)$$

Since $\phi(x)$ is integral,

$$F\left[\frac{f(x)}{(x-a)^2}\right]=F\left[\frac{f(x)}{x-a}\right]+F\left[\frac{f(a)}{(x-a)^2}\right], \tag{2}$$

where $F[\ ]$ represents the fractional part of the quotient found by performing the division indicated within the brackets.

When the two fractions in the second member of (2) are added, the numerator of the resulting fraction equals the numerator of the fraction in the first member.

Hence,

$$R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)R\left[\frac{f'(x)}{x-a}\right]+R\left[\frac{f'(a)}{(x-a)^2}\right],$$

where $R[\ ]$ represents the fractional part of the remainder found by performing the division indicated within the brackets.

$$\therefore R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)f'(a)+f(a). \tag{a}$$

Differentiating (1) and solving for $\frac{f(x)}{(x-a)^3}$,

$$\frac{f(x)}{(x-a)^3}=-\frac{f''(x)}{2!(x-a)}+\frac{f'(x)}{(x-a)^2}+\frac{f(a)}{(x-a)^3}+\frac{\phi''(x)}{2!}. \tag{3}$$

$$\therefore F\left[\frac{f(x)}{(x-a)^2}\right]=-F\left[\frac{f''(x)}{2!(x-a)}\right]+F\left[\frac{f'(x)}{(x-a)^2}\right]+F\left[\frac{f(a)}{(x-a)^3}\right]$$

and $$R\left[\frac{f(x)}{(x-a)^3}\right]=-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{x-a}\right]+(x-a)R\left[\frac{f'(x)}{(x-a)^2}\right]+R\left[\frac{f(a)}{(x-a)^3}\right].$$

Since $R\left[\frac{f''(x)}{x-a}\right]=f''(a)$, $R\left[\frac{f(a)}{(x-a)^3}\right]=f(a)$, and $R\left[\frac{f'(x)}{(x-a)^2}\right]$

$$=(x-a)f''(a)f'(a),$$

$$R\left[\frac{f(x)}{(x-a)^3}\right]=\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a). \tag{b}$$

Equations (a) and (b) prove the theorem true when $m=2$ or 3. We will now assume it true for $m\leq r$ and prove it true for $m=r+1$, or that

$$R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=\frac{1}{r!}(x-a)^r f^r(a)+R\left[\frac{f(x)}{(x-a)^r}\right].$$

By mathematical induction,

$$\frac{f(x)}{(x-a)^m}=\frac{(-1)^m f^{m-1}(x)}{(m-1)!(x-a)}+\frac{(-1)^{m-1}f^{m-2}(x)}{(m-2)!(x-a)^2}+\ldots.$$

$$+\frac{f'(x)}{(x-a)^{m-1}}+\frac{f(a)}{(x-a)^m}+\frac{(-1)^{m-1}\phi^{m-1}(x)}{(m-1)!}. \qquad (4)$$

If we let $m=r$ and $m=r+1$ in (4), we derive

$$R\left[\frac{f(x)}{(x-a)^r}\right]=R\left[\frac{f(a)}{(x-a)^r}\right]+(x-a)R\left[\frac{f'(x)}{(x-a)^{r-1}}\right]-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+$$

$$\ldots(-1)^r\frac{(x-a)^{r-1}}{(r-1)!}R\left[\frac{f^{r-1}(x)}{x-a}\right], \qquad (5)$$

and $R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=R\left[\frac{f(a)}{(x-a)^{r+1}}\right]+(x-a)\,R\left[\frac{f'(x)}{(x-a)^r}\right]$

$$-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+\ldots(-1)^{r+1}\frac{(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]. \qquad (6)$$

The second member of (6) can be derived from the second member of (5) by dividing the fraction in the bracket of each term by $(x-a)$ and adding the term $\frac{(-1)^{r+1}(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]$.

Corresponding to the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]$ in (5), we have the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]$ in (6).

Since the value of m in each bracket of the second member of (6) is less than $r+1$, and since for any value of $m\leq r$ the remainder can be found from the remainder for the preceding value of m by adding a term of degree one less than the value of m, therefore,

$$(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]=(-1)^{s-1}\frac{(x-a)^s}{s!}\left(\frac{(x-a)^{r-s}}{(r-s)!}f^{r-s+1}(a)\right.$$

$$\left.+R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]\right)=\frac{(-1)^{s-1}}{s!}\frac{(x-a)^r}{(r-s)!}f^r(a)+(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right].$$

In (6), the term $\frac{(-1)^{r-1}(x-a)^r}{r!} R\left[\frac{f^r(x)}{x-a}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a).$

Hence, $R\left[\frac{f(x)}{(x-a)^{r+1}}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a) + \sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a)$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)$$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a) + R\left[\frac{f(x)}{(x-a)^r}\right].$$

It remains to be proved that

$$(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a) + \sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}(x-a)^r}{s!\,(r-s)!}f^r(a) = \frac{(x-a)^r}{r!}f^r(a)$$

$$\text{or that } \frac{(-1)^{r-1}}{r!} + \sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!(r-s)!} = \frac{1}{r!}. \tag{7}$$

The factorial series $\sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!\,(r-s)!} = \frac{1}{(r-1)!} - \frac{1}{2!(r-2)!} + \frac{1}{3!(r-3)!} -$

$$\cdots\cdots \frac{(-1)^{r-2}}{(r-2)!2!} + \frac{(-1)^r}{(r-1)!}. \tag{8}$$

Multiplying (8) by $r!/r!$ and adding and subtracting $1/r!$, the series

$$= \frac{1}{r!} + \frac{1}{r!}\left[(r-1) - \frac{r(r-1)}{2!} + \frac{r(r-1)\,(r-2)}{3!} + \ldots\ (r-1)\ \text{terms}\right].$$

The sum of the terms in the bracket $= (-1)^r\frac{(r-1)!}{(r-1)!} = +1$ if r be even and -1 if r be odd.

Hence, (7) is true for all integral values of r.

Since we have proved the theorem true for $m=2$ and 3, it must be true for $m=4$, 5, 6, and for any integral value of m.

It will be observed that the limiting value of the remainder is Taylor's expansion of the function.

Also solved by H. V. Spunar and G. B. M. Zerr.

294. Proposed by O. L. CALLECOT, Gettysburg, S. Dak.

Find the limit of $\sum_{n=1}^{n=\infty} \frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)}$.

Solution by H. V. SPUNAR, C. E., East Pittsburg, Pa., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

$$\sum_{n=1}^{n=n} \frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)} = \sum_{n=1}^{n=n}\left[\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$= \Sigma\frac{1}{n} - \Sigma\frac{1}{n+1} + \Sigma\frac{1}{n+2} - \Sigma\frac{1}{n+3}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots$$

$$- \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \ldots$$

$$+ \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots$$

$$- \frac{1}{4} - \ldots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \ldots$$

$$= 1 + \tfrac{1}{3} = 1\tfrac{1}{3}.$$

Also solved by C. E. White and G. B. M. Zerr.

In order for the solution of this problem to be rigorous, the matter of convergency must be investigated. The equality

$$\sum_{n=1}^{n=n}\left[\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3}\right] = \sum_{n=1}^{n=\infty}\frac{1}{n} - \sum_{n=1}^{n=\infty}\frac{1}{n-1} + \sum_{n=1}^{n=\infty}\frac{1}{n+2} - \sum_{n=1}^{n=\infty}\frac{1}{n+3}$$

assumed to hold in the above solution is not always true.

GEOMETRY.

328. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

A sphere with the radius R is divided into two segments by a plane passed through it half way between the center and circumference. The smaller segment is divided into two parts by a plane passed through it at right angles to the base and cutting it half way between its center and circumference. Required the contents of the two parts of the segment.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and BENJAMIN F. FINKEL, Ph. D., Drury College, Springfield, Mo.

The equation of the sphere is $x^2+y^2+z^2=R^2$. The volume of $C-HIK$ is

$$V_1=\int dx \int dy \int dz=\int_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R} dx\, 2\int_0^{\sqrt{(\frac{3}{4}R^2-x^2)}} dy \int_{\frac{1}{2}R}^{\sqrt{(R^2-x^2-y^2)}} dz$$

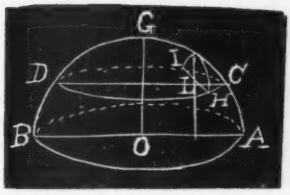
$$=\int_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R} dx\, 2\int_0^{\sqrt{(\frac{3}{4}R^2-x^2)}} [\sqrt{(R^2-x^2-y^2)}-\tfrac{1}{2}R]\,dy$$

$$=\int_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R} 2\left[\tfrac{1}{2}y\sqrt{(R^2-x^2-y^2)}+\tfrac{1}{2}(R^2-x^2)\sin^{-1}\frac{y}{\sqrt{(r^2-x^2)}}-\tfrac{1}{2}Ry\right]_0^{\sqrt{(\frac{3}{4}R^2-x^2)}} dx$$

$$=\int_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R} \left[(R^2-x^2)\cos^{-1}\left(\frac{\frac{1}{2}R}{\sqrt{(R^2-x^2)}}\right)-\tfrac{1}{2}R\sqrt{(\tfrac{3}{4}R^2-x^2)}\right]dx$$

$$=\Bigg[\,(R^2x-\tfrac{1}{3}x^3)\cos^{-1}\left(\frac{\frac{1}{2}R}{\sqrt{(R^2-x^2)}}\right)$$

$$-\tfrac{1}{24}R\int\left(\frac{9R^4-33R^2x^2+16x^4}{(R^2-x^2)\sqrt{(\frac{3}{4}R^2-x^2)}}\,dx\right)\Bigg]_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R}$$



$$=\Bigg[\,(R^2x-\tfrac{1}{3}x^3)\cos^{-1}\frac{\frac{1}{2}R}{\sqrt{(R^2-x^2)}}$$

$$-\tfrac{1}{24}R\int\left(\frac{17R^2}{\sqrt{(\frac{3}{4}R^2-x^2)}}-\frac{16x^2}{\sqrt{(\frac{3}{4}R^2-x^2)}}-\frac{8R^4}{(R^2-x^2)\sqrt{(\frac{3}{4}R^2-x^2)}}\right)dx\Bigg]_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R}$$

$$=\Bigg[\,(R^2x-\tfrac{1}{3}x^3)\cos^{-1}\frac{\frac{1}{2}R}{\sqrt{(R^2-x^2)}}-\tfrac{1}{24}R\left(11R^2\sin^{-1}\left(\frac{x}{\frac{1}{2}\sqrt{3}R}\right)\right.$$

$$\left.-16R^2\sin^{-1}\frac{x}{\sqrt{3}(R^2-x^2)}+8x\sqrt{(\tfrac{3}{4}R^2-x^2)}\right]_{\frac{1}{4}\sqrt{3}R}^{\frac{1}{2}\sqrt{3}R}$$

$$=R^3\left[\tfrac{13}{72}\pi-\tfrac{15}{64}\sqrt{3}\cos^{-1}\frac{2}{\sqrt{13}}-\tfrac{3}{8}\sin^{-1}\frac{1}{\sqrt{13}}+\tfrac{1}{16}\sqrt{3}\right].$$

The volume of $D-HIK$ is

$$V_2=4\int_{\frac{1}{2}R}^{R} dx \int_0^{\sqrt{(R^2-x^2)}} dy \int_0^{\sqrt{(R^2-x^2-y^2)}} dx - V_1=\tfrac{5}{24}\pi R^3-V_1$$

$$=R^3\left[\frac{\pi}{36}+\tfrac{15}{64}\sqrt{3}\cos^{-1}\left(\frac{2}{\sqrt{13}}\right)+\tfrac{3}{8}\sin^{-1}\left(\frac{1}{\sqrt{13}}\right)-\tfrac{1}{16}\sqrt{3}\right].$$

Also solved with different results by H. V. Spunar and J. Scheffer.

329. Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

1. Determine the equation of the locus of a fixed point in a circle of radius r, rolling along the axis of an upright cylinder of the same radius, while the axis revolves (carrying the circle with it) through an angle equal to the central angle of the rolling circle formed by the radii to the fixed point and the point of contact.

2. Suppose the point projected into the surface of the cylinder.

3. What is the surface generated by the radius of the rolling circle?

4. What is the surface generated by a radius of the cylinder through the moving point?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let a=the distance between the axis of the cylinder and the line about which it revolves. Also suppose the plane of the circle is determined by the line a and the axis of the cylinder. Let the circle, in its initial position, be in the xz plane with the point of contact in the same plane.

The position of the moving point at any time, using the axis of the cylinder as initial line and initial point of contact as origin, is $r\theta - r\sin\theta$, and $r(1-\cos\theta)$. We use the double sign, plus when the circle is not between the z-axis and the axis of the cylinder, and the minus when it is.

1. The locus is determined by $x=[a\pm r(1-\cos\theta)]\cos\theta$, $y=[a\pm r(1-\cos\theta)]\sin\theta$, $z=r\theta-r\sin\theta$, θ=central angle of circle.

2. If point is projected in surface of cylinder in plane of circle, the locus is determined by $x=(a\pm r)\cos\theta$, $y=(a\pm r)\sin\theta$, $z=r(\theta-\sin\theta)$.

3. Coordinates of the center of gravity of the moving radius of the circle are, $x=[a\pm r(1-\frac{1}{2}\cos\theta)]\cos\theta$, $y=[a\pm r(1-\frac{1}{2}\cos\theta)]\sin\theta$, $z=r(\theta-\frac{1}{2}\sin\theta)$.

$\therefore$ Surface$=rS$ where S is given as follows:

$$S=\int_0^\theta [4a^2\pm 8ar+8r^3\mp 4ar\cos\theta(4\sin^2\theta-1)+r^2(1-16\cos\theta)(1+\cos^2\theta)]^{\frac{1}{2}}\,d\theta.$$

4. Coordinates of center of gravity of radius of cylinder through moving point are $x=(a\pm\frac{1}{2}r)\cos\theta$, $y=(a\pm\frac{1}{2}r)\sin\theta$, $z=r(\theta-\sin\theta)$.

$\therefore$ Surface$=rS$ where

$$S=\int_0^\theta [(a\pm\tfrac{1}{2}r)^2+r^2(1-\cos\theta)^2]^{\frac{1}{2}}\,d\theta.$$

Also solved by H. V. Spunar.

CALCULUS.

257. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

If $A=\int_0^\infty \frac{dx}{\sqrt{x}\,(2a+x)^n}$, $B=\int_0^\infty \frac{y^n dy}{\sqrt{y}\,(a^2+y^2)^n}$, find A/B.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let $x=2a\tan^2\theta$, $y=a\tan\phi$, $m=\frac{1}{2}(2n-1)$.

$$\therefore A=\frac{2}{(2a)^{n-\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\cos^{2(n-1)}\theta\,d\theta=\frac{1}{2(2a)^m}\int_0^{\frac{1}{2}\pi}\cos^{2m-1}\theta\,d\theta=\frac{\sqrt{(\pi)}\,\Gamma(m)}{(2a)^m\Gamma\left(\frac{2m+1}{2}\right)}.$$

$$B=\frac{1}{a^{n-\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\tan^{n-\frac{1}{2}}\phi\cos^{2(n-1)}\phi\,d\phi=\frac{1}{a^m}\int_0^{\frac{1}{2}\pi}\sin^m\phi\cos^{m-1}\phi\,d\phi$$

$$=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}{2a^m\,\Gamma\left(\frac{2m+1}{2}\right)}.$$

But $\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m}{2}\right)=\frac{\sqrt{(\pi)}\,\Gamma(m)}{2^{m-1}}$.

$$\therefore B=\frac{\sqrt{(\pi)}\,\Gamma(m)}{(2a)^m\Gamma\left(\frac{2m+1}{2}\right)}=A.\quad\text{Hence } A/B=1.$$

Exhaustive solutions, though differing in results from the one given here, were received from M. V. Spunar, Francis Rust, and T. G. Wodo.

258. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Evaluate $\int_0^{\frac{1}{2}a}\frac{dx}{\sqrt{[2ax-x^2\sqrt{(a^2-x^2)}]}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $x=x\sin\theta$. Then

$$I=\int_0^{\frac{1}{2}a}\frac{dx}{\sqrt{[2ax-x^2\sqrt{(a^2-x^2)}]}}=\int_0^{\frac{1}{2}\pi}\frac{\cos\theta\,d\theta}{\sqrt{[2\sin\theta-a\sin^2\theta\cos\theta]}}.$$

$$\therefore I=\frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi}\left[1+\frac{a}{4}\sin\theta\cos\theta+\frac{3a^2}{32}\sin^2\theta\cos^2\theta+\frac{5a^3}{128}\sin^3\theta\cos^3\theta\right.$$

$$\left.+\frac{35a^4}{2048}\sin^4\theta\cos^4\theta+\frac{64a^5}{8192}\sin^5\theta\cos^5\theta+\frac{231a^6}{65536}\sin^6\theta\cos^6\theta+\ldots\right]\frac{\cos\theta\,d\theta}{\sqrt{(\sin\theta)}}$$

$$=1+\frac{31}{15}\cdot\frac{a^2}{2^9}+\frac{85015}{1989}\cdot\frac{a^4}{2^{19}}+\frac{2350494}{38675}\cdot\frac{a^6}{2^{28}}+\ldots.+\frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi}\left[\frac{a}{4}\sin\theta\cos\theta\right.$$

$$+\frac{5a^3}{128}\sin^3\theta\cos^3\theta+\frac{63a^5}{8192}\sin^5\theta\cos^5\theta+\ldots\Big]\frac{\cos\theta\,d\theta}{\sqrt{(\sin\theta)}}.$$

$$\frac{1}{\sqrt{(\sin\theta)}}=(1-\cos^2\theta)^{-\frac{1}{4}}.$$

$$\therefore I=1+\frac{31}{15}\cdot\frac{a^2}{2^9}+\frac{85015}{1989}\cdot\frac{a^4}{2^{19}}+\frac{2350494}{38675}\cdot\frac{a^6}{2^{28}}+\ldots$$

$$+\frac{1}{\sqrt{2}}\Big[\frac{a}{35.2^5}\Big(\frac{341}{3}-\frac{20611\sqrt{3}}{2^9}+\ldots.\Big)+\frac{a^3}{21.2^8}\Big(\frac{41}{3}-\frac{2616\sqrt{3}}{2^{10}}+\ldots.\Big)$$

$$+\frac{a^5}{11.2^{10}}\Big(1-\frac{8181\sqrt{3}}{2^{14}}+\ldots.\Big)\Big]+\ldots.$$

a cannot be greater than $\frac{8}{3}\sqrt{3}$.

This solution, to be complete, should have investigated the matter of convergency and, since the function vanishes at the lower limit, also the condition of determinateness.

The proposer of 259 (256) suggests that the equation be changed to $(1+y+2axy)dx+x(1+x)dy=0$.

MECHANICS.

211. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A smooth elliptic wire, axis vertical, has a small ring sliding on it, connected by elastic strings with each focus. Either string is just unstretched when the ring is nearest the corresponding focus. The modulus of elasticity is W/n, where W oz. is the weight of the ring. Find the distance of the ring from the upper focus in the different positions of equilibrium, and in each case discuss the nature of the equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let F be the upper focus of the elliptic wire, semi-major axis a, eccentricity e. Let r, $2r-a$ be the lengths of the strings from the upper and lower foci to the ring at the point P on the curve; G the intersection of the normal from P with the axis major; θ the angles the strings make with the tangent at P; ϕ the angle the tangent at P makes with the major axis; T the tension of string r; Q the tension of string $2a-r$; m the weight of the ring and strings that cause a downward force due to gravity; $a(1-e)$=unstretched length of each string.

Then $PF=r$, $GF=er$, $\sin GPF=\cos\theta$, $\sin PGF=\cos\phi$, $\sin PGF:\sin GPF=r:er$. $\therefore\cos\phi/\cos\theta=1/e$.

Also $r=a(1-e)(1+Tn/W)$, $2a-r=a(1-e)(1+Qn/W)$.

$$\therefore T=\frac{W(r-a+ae)}{an(1-e)}, \quad Q=\frac{W(a+ae-r)}{an(1-e)}.$$

I. For equilibrium, $T\cos\theta=Q\cos\theta+m\cos\phi$.

$$\therefore \frac{T-Q}{m}=\frac{\cos\phi}{\cos\theta}=\frac{1}{e}. \quad \therefore \frac{2W(r-a)}{amn(1-e)}=\frac{1}{e}.$$

$$\therefore r=a+\frac{amn(1-e)}{2eW}=a+\frac{an(1-e)}{2e}, \text{ when } m=W.$$

When the ring is displaced it will tend to regain this same position of equilibrium.

II. For equilibrium, $T\cos\theta=Q\cos\theta=0$.

$\therefore r=a(1-e)$ and $r=a(1+e)$, or the upper and lower vertices. When the ring is displaced from either of these positions it will tend to equilibrium in I.

212. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

A peg A is vertically d feet above a peg B. A string AD, a feet long, with two equal, jointed rods DC, CB form the whole figure. Discuss the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let θ=the angle the string makes with the vertical; ϕ=the angle DC makes with the vertical; ψ=the angle CB makes with the vertical; b=the length of each rod; W its weight. Also regard the string as weightless, and let x=the depth of the center of gravity of the system below A.

$$\therefore x=[w(a\cos\theta+\tfrac{1}{2}b\cos\phi)+w(d\pm\tfrac{1}{2}b\cos\psi)]/2w\ldots(1).$$

Projecting vertically, we get $a\cos\theta+b\cos\phi\mp b\cos\psi=d\ldots(2)$.

Also, $a^2+d^2-2ad\cos\theta=2b^2-2b^2\cos(\psi-\phi)\ldots(3)$.

$a\cos\theta$ from (2) in (1) and (3) gives $x=[w(2d-\tfrac{1}{2}b\cos\phi\pm\tfrac{3}{2}b\cos\psi)]/2w\ldots(4)$.

$$a^2-d^2+2db\cos\phi\mp 2db\cos\psi=2b^2-2b^2\cos(\psi-\phi)\ldots(5).$$

Differentiating (4) and (5), we get $3\sin\psi\, d\psi=\pm\sin\phi\, d\phi\ldots(6)$.

$$[b\sin(\psi-\phi)\pm d\sin\psi]d\psi=[b\sin(\psi-\phi)-d\sin\phi]d\phi\ldots(7).$$

Eliminating dx and $d\phi$ between (6) and (7),

$$b\sin(\psi-\phi)(3\sin\psi\pm\sin\phi)=2d\sin\phi\sin\psi\ldots(8).$$

(5) and (8) determine the equilibrium. The $\pm$ sign is used as follows: if a is long enough to permit C to fall below B use the upper sign; if not, use the lower.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

No solution received.

149. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that every prime of the form $4n+1$ may be expressed as the sum of two parts r and s such that r^2+rs+s^2+1 is divisible by the prime.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Let $p=r+s$=a prime of the form $4n+1$. Substitute $s=p-r$ in $r^2+rs+s^2+1\equiv 0 \pmod{p}$ and we have $r^2+r(p-r)+(p-r)^2+1\equiv 0 \pmod{p}$ or $r^2+1\equiv 0 \pmod{p}$, which is always possible when p is a prime of the form $4n+1$, and in many other cases.

Example. If $p=5$, $r^2+1\equiv 0 \pmod{5}$ has the solution $r\equiv 2 \pmod{5}$. $\therefore r=2$, $s=3$.

II. Solution by E. T. BELL, University of Washington.

When $p\equiv 1 \pmod{4}$: $\left(\frac{-1}{p}\right)=1$ (Legendre's symbol).

Hence, for r a root of $x^2\equiv -1 \pmod{p}$, $rp-(r^2+1)\equiv 0 \pmod{p}$; whence, putting $s=p-r$, $rs-1\equiv 0 \pmod{p}$.

Therefore, $p^2-(rs-1)\equiv 0 \pmod{p}$; or, r^2+rs+s^2+1 is a multiple of p; which establishes the stated theorem.

III. Solution by W. F. KING, Ottawa, Canada.

Since $r+s$ is a prime of form $4n+1$, it may be resolved in one way into the sum of two squares. Suppose it to be so resolved, and to be equal to h^2+k^2.

By the usual process, express h/k as a continued fraction in the form

$$\frac{h}{k}=x+\frac{1}{x_1+}\,\frac{1}{x_2+}\,\frac{1}{x_3+}\cdots$$

Form the successive convergents, and let the convergent next before the last (h/k), be (a/b).

Then $ak-bh=\pm 1$.

And $(ah+bk)^2+(ak-bh)^2=a^2h^2+b^2k^2+a^2k^2+b^2h^2=(a^2+b^2)(h^2+k^2)$.

Therefore, $(ah+bk)^2+(ak-bh)^2$, or $(ah+bk)^2+1$, is divisible by h^2+k^2, or $r+s$.

Now $a<h$, $b<k$. $\therefore ah+bk<h^2+k^2$.

Calling $ah+bk$, r, we have r^2+1, and therefore $r^2+1+s(r+s)$, or r^2+rs+s^2+1, divisible by $r+s$.

It is evident that the number $ah+bk$ which is less than h^2+k^2 and is such that $(ah+bk)^2+1$ is divisible by h^2+k^2, can always be found.

Also solved by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

303. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate the determinant

$$\begin{vmatrix} D_1 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ x_1x_2 & D_2 & x_2x_3 & \dots & x_2x_n \\ x_1x_3 & x_2x_3 & D_3 & \dots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & \dots & D_n \end{vmatrix}$$

304. Proposed by C. N. SCHMALL, New York City.

A policeman on a motor-cycle starts in pursuit of an automobile when the latter has a headway of $\frac{1}{2}$ a mile. A pedestrian who is $\frac{1}{4}$ of a mile ahead of the auto and who is walking at the rate of 5 miles an hour, notices that when the auto overtakes him the policeman is only $\frac{5}{12}$ of a mile behind the auto, and $2\frac{1}{2}$ miles from where the officer started; he overtakes the auto. How long did the chase last?

305. Proposed by S. A. COREY, Hiteman, Iowa.

Prove or disprove, that $\sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2+4} = \frac{\pi}{4} - \frac{\pi}{8}\left(\frac{\cosh \pi}{\sinh \pi}\right)$.

GEOMETRY.

336. Proposed by F. H. HODGE, The University of Chicago.

A man owning a rectangular field b=300 feet by a=600 feet, wishes to lay out driveways of equal width having the diagonals of the field as center lines and such that the area of the driveways shall be n/m=one-half, of the area of the field. Determine the width of the driveways.

337. Proposed by T. N. HILDEBRANT, The University of Chicago.

Required the locus of the vertices of the parabolae having a given focus and passing through a given point.

338. Proposed by C. N. SCHMALL, 239 East 7th Street, New York.

Given the base and vertical angle of a triangle, find the locus of the center of its "nine-point" circle. [Ex. 28, p. 65, Casey's *Sequel to Euclid*.]

CALCULUS.

263. Proposed by V. M. SPUNAR, M. S., C. E., East Pittsburg, Pa.

Find a point such that the sum of the squares of its distances from n given points shall be a minimum, and prove that the value so found is $1/n$th part of the sum of the squares of the mutual distances between the n points, taken two and two.

264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The join of the center of curvature of a curve to the origin is at α to the Initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d^2\psi}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

MECHANICS.

219. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A rod length $a\sqrt{3}$, weight W, has at each end a smooth ring which can slide on a vertical circle radius r. Each ring is attached by an elastic string (natural lengths a, b; moduli μa, μb) to the highest point of the circle. Find the inclination of the rod to the horizon in a position of equilibrium.

220. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Four particles A, B, C, D lie on a smooth table at the corners of a rhombus. AB, BC, CD, DA are light inextensible strings connecting the particles. The angle at A is acute. A blow is given to A along the diagonal, away from C. Find the ratio of the initial velocity of C to that of A.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

155. Proposed by PROF. R. D. CARMICHAEL, Anniston, Alabama.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

156. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for a, b, c, d, and e in the equation, $a^2+b^2+c^2+d^2=e^2$.

AVERAGE AND PROBABILITY.

199. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a given square. Two points are taken at random within the square but without the circle. What is the chance the distance between the points does not exceed the side of the square?

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ON DeMOIVRE'S QUINTIC.

By DR. R. L. BORGER, University of Illinois.

§1. For the domain of rational numbers, DeMoivre's quintic

$$x^5+px^3+\tfrac{1}{5}p^2x+r=0, \tag{1}$$

for values of p and r making the discriminant

$$\Delta\equiv\left(\frac{r}{2}\right)^2+\left(\frac{p}{5}\right)^5\neq 0,$$

will be shown to have as its Galois group *either the metacyclic group G_{20} or a cyclic group C_4*. We may then readily deduce the following properties:

DeMoivre's quintic is solvable by radicals.

Either all the roots are real or only one root is real.

Not more than one root is rational; if the equation is reducible in $R(1)$, its left member is the product of a linear and an irreducible quartic factor.

If the equation is irreducible in $R(1)$, any root is a rational function of an arbitrary pair of roots.

To determine the Galois group of (1), we make use on the one hand of Cayley's resolvent sextic for any quintic, and on the other hand of the following lemma:*

If we know a rational function of the roots of an algebraic equation $f(x)=0$ having the properties:

(*i*) That it is formally invariant under the substitutions of a group G' and under no others.

(*ii*) That it has a value in the domain of rationality.

(*iii*) That it is distinct from its conjugates under the substitutions of the symmetric group $G_n!$, then the Galois group G of $f(x)=0$ is a subgroup of G'.

* Dickson, *Algebraic Equations*, p. 59, §65.